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PHASE TRANSITIONS IN NONHOMOGENEOUS, AGING, VISCOELASTIC BODIES

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Abstract—We propose a thermodynamic model for a heterogeneous, aging, viscoelastic medium at infinitesimal strains and analyze the main thermodynamic potentials. We find the conditions for the relaxation measure, which guarantee the Second Law of thermodynamics to be satisfied. Variational principles for viscoelastic bodies at isothermal and adiabatic loading are formulated. The energy ratios for the determination of the interface at the phase transition are derived. We analyze the local melting process in the viscoelastic body at isothermal loading and study the rheology influence on the development of a spherical liquid phase nucleus.

1. INTRODUCTION

This paper is devoted to the construction of thermodynamic potentials in the linear viscoelasticity of aging media. We need a thermodynamic model to study (1) the stability of the body, (2) the displacement of the interface during phase transitions in solids, (3) the extension of cracks in the medium and (4) the numerical analysis of the stress fields. The thermodynamics of materials with memory at finite strains was investigated by Coleman (1964). Trapeznikov (1978) proposed a thermodynamic model of an aging viscoelastic medium. Another approach to this problem was developed by Arutyunyan *et al.* (1987). The defect of these potentials is the necessity to introduce supplementary time-dependent functions or free-energy kernels, which are difficult to examine experimentally. The expressions for the free energy of nonaging viscoelastic media were considered by Dafermos (1970a,b).

In the second part of the paper we formulate the variational principles for viscoelastic media at isothermal and adiabatic loading. Our formulations are different from the standard variational principles in elasticity (Landau and Lifshitz, 1976; Berdichevskii, 1983a) and viscoelasticity (Rabotnov, 1979) because we consider the complete deformation process and not just the initial and final states of the system. Then we discuss the connection between introduced variational principles and the phenomenological approach based on the concept of "availability criteria" (Alts and Strehlow, 1984; Strehlow, 1988). We show that the equilibrium equation and the stress boundary condition can be deduced from these variational principles. Moreover, using the variational principles we obtain the energy ratio on the interface at the first-order phase transitions in a viscoelastic medium. The interface conditions for the phase transformation in an elastic body at infinitesimal strains were found by Roitburd (1971). The phase transitions in elastic bodies at finite strains have been studied by Grinfel'd (1980), Gurtin (1983), Kondaurov and Nikitin (1983) and Fonseca (1987).

In the third part of this paper we apply the obtained results to the problem of local melting. We analyze the growth of the spherical liquid nucleus in aging viscoelastic media at isothermal loading. The aging influence on the growth of the nucleus is analyzed. The creation of a spherical nucleus in elastic media has been considered by Lifshitz and Gulida (1952). Another approach to this problem was proposed by Johnson and Voorhees (1987). The creation of an ellipsoidal nucleus of melt in an elastic body was investigated by Berdichevskii (1983b). The techniques used in these papers are founded on the hypothesis that the nuclei are generated instantaneously and cannot describe the growth of these nuclei. The growth of nuclei in viscoelastic bodies during phase transformation was studied by Arutyunyan and Drozdov (1985).

2. THERMODYNAMIC POTENTIALS FOR AGING VISCOELASTIC MEDIA

The body is in its natural state at temperature $\theta = \theta_0$ and occupies the connected domain Ω with the smooth boundary Γ. Let us introduce the coordinate system $\xi = (\xi_i)$ in the domain Ω. At the moment $t = 0$ body forces $F(t, \xi)$ and surface forces $f(t, \xi)$ are applied to the body. We neglect the inertia forces and assume that the strains are infinitesimal. We denote the displacement vector by $u(t, \xi)$ and the corresponding strain tensor by $\varepsilon(t, \xi)$:

$$\varepsilon = (\nabla u + \nabla u^{\mathrm{T}})/2. \tag{1}$$

The mechanical behaviour of the material obeys the constitutive laws of the aging homogeneous and isotropic, linear viscoelastic body with elastic dilatation. We suppose that the thermal expansion can be neglected and write the following expressions for the free energy per unit mass Ψ

$$\Psi = \Psi_0 + W - H_0(\theta - \theta_0) - c(\theta - \theta_0)^2/(2\theta_0),$$

$$\rho W(t) = \tfrac{1}{2}K\bar{\varepsilon}^2(t) + G(t)\left[R(t,0)e(t) : e(t) + \int_0^t \frac{\partial R}{\partial \tau}(t,\tau)(e(t) - e(\tau)) : (e(t) - e(\tau))\,\mathrm{d}\tau\right],$$

$$\bar{\varepsilon} = \varepsilon : I, \quad e = \varepsilon - \tfrac{1}{3}\bar{\varepsilon}I, \quad R(t,\tau) = G^{-1}(t)[G(\tau) + Q(t,\tau)], \tag{2}$$

where Ψ_0, H_0 are the free energy and entropy per unit mass in the natural configuration at the temperature θ_0, ρ is the mass density, W denotes the mechanical energy per unit mass, $c > 0$ is the heat capacity per unit mass, $K > 0$ is the constant bulk modulus, $G(t) > 0$ is the current shear modulus, $Q(t, \tau)$ is the relaxation measure, I is the unit tensor. We omit the argument ξ for shortening.

Expressions for the relaxation measures of some viscoelastic materials are represented in Arutyutyan and Kolmanovskii (1983). The function $R(t, \tau)$ is continuously differentiable and satisfies the conditions:

$$R(t,t) = 1, \quad R(t,\tau) \geqslant 0, \quad \frac{\partial R}{\partial \tau}(t,\tau) \geqslant 0, \quad (0 \leqslant \tau \leqslant t). \tag{3}$$

Identity (3) defines the norm of the relaxation measure and is accepted here for convenience. The inequalities (3) guarantee the convexity of the specific mechanical energy in the natural configuration.

Equality (2) is the Taylor expansion of the specific free energy Ψ when third order terms are neglected [see for example, Arutyunyan *et al.* (1987)].

The entropy per unit mass $H = -\partial\Psi/\partial\theta$ is given by the formula:

$$H = H_0 + c(\theta - \theta_0)/\theta_0. \tag{4}$$

The internal energy Φ per unit mass is calculated using the formula $\Phi = \Psi + H\theta$. With the aid of (2) and (4) we obtain

$$\Phi = \Phi_0 + W + c(\theta^2 - \theta_0^2)/(2\theta_0), \quad (\Phi_0 = \Psi_0 + H_0\theta_0). \tag{5}$$

Consider a heterogeneous, aging body, in which different elements are created at different moments of time [see Arutyunyan and Kolmanovskii (1983)]. Denote by $\tau^*(\xi) \leqslant 0$ the time when the material element is generated in the neighbourhood of the point ξ. We write the following expression for the specific mechanical energy of a nonhomogeneous, aging, viscoelastic medium:

$$\rho W(t) = \tfrac{1}{2}K\tilde{\varepsilon}^2(t) + G(t-\tau^*)\Big[R(t-\tau^*, -\tau^*)e(t):e(t) + \int_0^t \frac{\partial R}{\partial \tau}(t-\tau^*, \tau-\tau^*)(e(t)-e(\tau)):(e(t)-e(\tau))\,d\tau\Big]. \quad (6)$$

3. VARIATIONAL PRINCIPLES IN THE THEORY OF VISCOELASTICITY

Usually two types of loading processes are considered: isothermal and adiabatic loading. We first deal with the isothermal case, when the body temperature is constant and equal to θ_0. Let the time $t \geqslant 0$ and the deformation history $\{u(\tau, \xi), (0 \leqslant \tau \leqslant t)\}$ be fixed. Denote by $U(t)$ the set of displacement fields $u^0(t, \xi)$, which are continuously differentiable in Ω. Choose an admissible field $u^0(t, \xi) \in U(t)$ and a corresponding strain field $\varepsilon^0(t, \xi)$. Here and below we denote by the symbol with index "0" any admissible value of thermodynamic quantity. The value of such a quantity, which is realized in the loading process is denoted by the same symbol without index "0".

From (2) and (6) it follows that Helmholtz free energy of the body is equal to

$$\Psi(t) = \int_\Omega \rho\Psi(t)\,dv = \rho\Psi_0|\Omega| + \bar{W}(t), \quad |\Omega| = \int_\Omega dv, \quad \bar{W}(t) = \int_\Omega \rho W(t)\,dv,$$

$$\rho W(t) = \tfrac{1}{2}K[\tilde{\varepsilon}^0(t)]^2 + G(t-\tau^*)\Big[R(t-\tau^*, -\tau^*)e^0(t):e^0(t) + \int_0^t \frac{\partial R}{\partial \tau}(t-\tau^*, \tau-\tau^*)(e^0(t)-e(\tau)):(e^0(t)-e(\tau))\,d\tau\Big],$$

$$\tilde{\varepsilon}^0 = \varepsilon^0 : I, \quad e^0 = \varepsilon^0 - \tfrac{1}{3}\tilde{\varepsilon}^0 I, \quad (7)$$

where dv is a volume element and ds is a surface element.

Define the work of external forces by the formula

$$A(t) = A_-(t) + \int_\Omega \rho F(t)\cdot(u^0(t)-u_-(t))\,dv + \int_\Gamma f(t)\cdot(u^0(t)-u_-(t))\,ds, \quad (8)$$

where symbol $X_-(t)$ denotes the limit of the function $X(\tau)$ as $\tau \to t-0$.

By virtue of Lurie (1980), the free energy of the system E is defined by

$$E(t) = \Psi(t) - A(t). \quad (9)$$

Now we may postulate the principle of minimum free energy of a system:

The real displacement field $u(t, \xi)$ minimizes the functional $E(t)$ on the set $U(t)$.

We will show that the equilibrium equation

$$\nabla\cdot\sigma(t) + \rho F(t) = 0 \quad (10)$$

and the boundary condition on the surface Γ

$$N\cdot\sigma(t) = f(t), \quad (\xi \in \Gamma) \quad (11)$$

follow from this principle using the constitutive laws of a nonhomogeneous viscoelastic media

$$\bar{\sigma}(t) = 3K\bar{\varepsilon}(t), \quad s(t) = 2G(t-\tau^*)\left[e(t) - \int_0^t \frac{\partial R}{\partial \tau}(t-\tau^*, \tau-\tau^*)e(\tau)\,d\tau\right]. \tag{12}$$

Here N is the outward unit normal vector to Γ, $\sigma(t,\xi)$ is the stress tensor, $\bar{\sigma} = \sigma : I$, $s = \sigma - \frac{1}{3}\bar{\sigma}I$.

Proof. Let the admissible variation of the displacement vector be $\delta u(t,\xi)$ and the corresponding variation of the strain tensor be $\delta\varepsilon(t,\xi)$. The variation of the function $E(t)$ is equal to

$$\delta E(t) = \int_\Omega \left\{K\bar{\varepsilon}(t)\delta\bar{\varepsilon}(t) + 2G(t-\tau^*)\left[R(t-\tau^*, -\tau^*)e(t) + \int_0^t \frac{\partial R}{\partial \tau}(t-\tau^*, \tau-\tau^*)(e(t)-e(\tau))\,d\tau\right] : \delta e(t)\right\} dv$$

$$- \int_\Omega \rho F(t)\cdot\delta u(t)\,dv - \int_\Gamma f(t)\cdot\delta u(t)\,ds.$$

From this equality, (3) and (12) it follows that

$$\delta E(t) = \int_\Omega [\sigma(t) : \delta\varepsilon(t) - \rho F(t)\cdot\delta u(t)]\,dv - \int_\Gamma f(t)\cdot\delta u(t)\,ds. \tag{13}$$

Using (1), (13) and Stoke's formula we obtain

$$\delta E(t) = -\int_\Omega [\nabla\cdot\sigma(t) + \rho F(t)]\cdot\delta u(t)\,dv + \int_\Gamma [N\cdot\sigma(t) - f(t)]\cdot\delta u(t)\,ds. \tag{14}$$

Hence the equalities (10) and (11) follow from (14) as necessary conditions of being a minimum of the function $E(t)$ because of the arbitrariness of the field $\delta u(t,\xi)$.

Now we consider adiabatic loading, i.e. the heat flow to the body vanishes. In addition we assume that the characteristic time for changing the external forces and the characteristic time for relaxing the stresses exceed the characteristic time which is necessary to establish thermodynamic equilibrium. In this case the body is in the thermodynamic equilibrium state at any $t \geqslant 0$ and the temperature θ does not depend on the coordinates.

Let the time $t \geqslant 0$ and the deformation history $\{\theta(\tau), u(\tau,\xi), (0 \leqslant \tau < t)\}$ be fixed. Denote by $\Theta(t)$ the set of admissible temperature values $\theta^0(t) \geqslant 0$. Let $B(t) = \Theta(t) \times U(t)$. Choose an admissible element $(\theta^0(t), u^0(t,\xi)) \in B(t)$. The internal energy of the body $\bar{\Phi}(t)$ and the entropy of the body $\bar{H}(t)$ are equal to

$$\bar{\Phi}(t) = \rho|\Omega|[\Phi_0 + c((\theta^0(t))^2 - \theta_0^2)/(2\theta_0)] + \bar{W}(t), \tag{15}$$

$$\bar{H}(t) = \rho|\Omega|[H_0 + c(\theta^0(t) - \theta_0)/\theta_0]. \tag{16}$$

The First Law of thermodynamics implies

$$\Phi(t) - \Phi_-(t) = A(t) - A_-(t). \tag{17}$$

We postulate Gibbs' principle:

The real displacement field $u(t,\xi)$ and the real temperature $\theta(t)$ maximize the entropy of the body $\bar{H}(t)$ on the subset $B(t)$, whose elements obey the conservation of energy (17).

Substituting (15) in (17) and using (9) we get

$$\theta^0(t) = \theta_-(t)\left[1 + \frac{2\theta_0}{\rho c|\Omega|\theta_-^2(t)}(E_-(t) - E(t))\right]^{1/2}.$$

This equality and (16) yield

$$\bar{H}(t) = \rho|\Omega|\left\{H_0 + c\left[\frac{\theta_-(t)}{\theta_0}\left(1 + \frac{2\theta_0}{\rho c|\Omega|\theta_-^2(t)}(E_-(t) - E(t))\right)^{1/2} - 1\right]\right\}. \tag{18}$$

According to (18) the displacement field $u(t, \xi)$, which maximizes the entropy of the body $\bar{H}(t)$ minimizes the free energy of the system $E(t)$. With the aid of (14) and (18) we discover that Gibbs' principle implies the equilibrium equation (10) in Ω and the boundary condition (11) on the surface Γ.

4. GIBBS' PRINCIPLE AND THE SECOND LAW OF THERMODYNAMICS

Let us consider the adiabatic loading of an elastic body ($G(t) = G = \text{const}$, $Q(t, \tau) = 0$) at time interval $[0, T]$. Divide this interval by points $t_k = k\Delta$, $\Delta = T/N$, $k = 0, 1, \ldots, N$. Introduce the piecewise constant external forces $F_N(t, \xi)$, $f_N(t, \xi)$ using the formulae

$$F_N(t, \xi) = F(t_k, \xi), \quad f_N(t, \xi) = f(t_k, \xi), \quad (t_k \leqslant t < t_{k+1}).$$

Suppose that there exists a constant $c_1 > 0$ such that for any $k \geqslant 0$

$$\|F(t_{k+1}) - F(t_k)\|_\Omega \leqslant c_1\Delta, \quad \|f(t_{k+1}) - f(t_k)\|_\Gamma \leqslant c_1\Delta, \tag{19}$$

where

$$\|F(t)\|_\Omega^2 = \int_\Omega F(t, \xi) \cdot F(t, \xi)\, dv, \quad \|f(t)\|_\Gamma^2 = \int_\Gamma f(t, \xi) \cdot f(t, \xi)\, ds.$$

Denote by $u_N(t, \xi)$ the displacement field corresponding to the forces $F_N(t, \xi)$, $f_N(t, \xi)$.

One can show that according to (19) there exists a constant $c > 0$ such that for any $t \in [0, T]$

$$\|u(t) - u_N(t)\|_\Omega \leqslant c\Delta. \tag{20}$$

Suppose that at time $t_k - 0$ the body is in thermodynamic equilibrium under the action of the forces $F(t_{k-1}, \xi)$, $f(t_{k-1}, \xi)$. This state is characterized by the displacement field $u_N(t_k, \xi)$ and the temperature $\theta_N(t_k)$. During the piecewise constant loading process the forces $F(t_k, \xi)$ and $f(t_k, \xi)$ are applied to the body at the moment t_k. Under the action of these forces the body leaves the equilibrium state. We assume that the body tends to move to another equilibrium state and that the characteristic time of the transition process τ_k is proportional to the intensity of the change in the external forces

$$\tau_k = c_2[\|F(t_k) - F(t_{k-1})\|_\Omega + \|f(t_k) - f(t_{k-1})\|_\Gamma]. \tag{21}$$

Here the constant $c_2 > 0$ depends on the mechanical properties of the material and the dimensions of the domain Ω only.

The loading process is called a continuous one if the characteristic time of the external forces changing exceeds the time which is necessary to establish thermodynamic equilibrium in the body:

$$\tau_k < \Delta. \tag{22}$$

From (19) and (21) it follows that the estimate (22) may be written as $c_1 c_2 < 1$. This inequality is a restriction only on the rate of change in the external load.

According to (22) under the piecewise constant loading process at the moment $t_{k+1} - 0$ the body is in a thermodynamic state of equilibrium which is characterized by the displacement field $u_N(t_{k+1}, \xi)$ and the temperature $\theta_N(t_{k+1})$.

Writing the equation of balance of energy and the Second Law of thermodynamics for the adiabatic loading process and by virtue of Strehlow (1988) we neglect the kinetic energy of the body to get

$$\Phi^*(t) = \int_\Omega \rho F_N(t) \cdot u_N^*(t)\, dv + \int_\Gamma f_N(t) \cdot u_N^*(t)\, ds, \tag{23}$$

$$\bar{H}^*(t) \geqslant 0. \tag{24}$$

Integrating (23) from t_k to $t \in [t_k, t_{k+1})$ we find

$$\Phi(t) - \Phi(t_k) = \int_\Omega \rho F(t_k) \cdot (u_N(t) - u_N(t_k))\, dv + \int_\Gamma f(t_k) \cdot (u_N(t) - u_N(t_k))\, ds. \tag{25}$$

From (24) it follows that the entropy of the body $\bar{H}(t)$ does not decrease in time. Hence, at the moment t_{k+1} the function $\bar{H}(t)$ has its maximum value on the set of the displacement fields $u_N(t, \xi)$ and the temperature fields $\theta_N(t, \xi)$, which are realized during the transition process and satisfy eqn (25).

According to Gibbs' principle, at the moment t_{k+1} the function $\bar{H}(t)$ has its maximum value on the set of admissible displacement fields $u^0(t_{k+1}, \xi)$ and temperature fields $\theta^0(t_{k+1})$, which satisfy (25) at $t = t_{k+1}$.

Note that Gibbs' principle does not follow from the Second Law of thermodynamics and vice versa. By the variational principle we find a maximum value of the function $\bar{H}(t)$ on the set of any admissible displacement field $u^0(t_{k+1}, \xi)$, which may be wider than the set of displacement fields realized during the transition process. On the other hand the heterogeneous temperature fields $\theta_N(t, \xi)$ may be realized during the transition process, but by formulating Gibbs' principle we consider the homogeneous temperature fields $\theta^0(t_{k+1})$ only.

We note the analogy between these assertions and the ergodic property of stochastic processes. By virtue of Gikhman and Skorokhod (1977), the stationary stochastic process is called ergodic if its mean value in time coincides with its mean value on the set of realizations. The Second Law of thermodynamics says that the entropy of the body has its maximum value in time at the interval $[t_k, t_{k+1})$ at a final equilibrium state. Gibbs' principle says that the real equilibrium state maximizes the entropy of the body on the set of admissible equilibrium states. Equivalence of these assertions means that nonequilibrium transition processes have the ergodic property [see Landau and Lifshitz (1976)].

By (20) we may proceed to the limit as $N \to \infty$ and get Gibbs' principle formulated in Section 2 for the continuous loading processes in elastic bodies. Existence of viscosity in the material does not change our formulation of Gibbs' principle if we suppose that the characteristic time of relaxation exceeds the characteristic time of changing of external forces Δ, because in this case we may neglect the creep of material during the transition process.

5. THERMODYNAMIC INEQUALITIES IN LINEAR VISCOELASTICITY

Let us consider the adiabatic loading of a nonhomogeneous, aging, viscoelastic body. Writing the First Law of thermodynamics in differential form we have

$$\Phi^{\cdot}(t) = \int_{\Omega} \rho F(t) \cdot u^{\cdot}(t)\, dv + \int_{\Gamma} f(t) \cdot u^{\cdot}(t)\, ds. \tag{26}$$

From (15), (16) and (26) it follows that for the process $\theta(t)$, $u(t, \xi)$:

$$\theta(t)\bar{H}^{\cdot}(t) = \int_{\Omega} \rho F(t) \cdot u^{\cdot}(t)\, dv + \int_{\Gamma} f(t) \cdot u^{\cdot}(t)\, ds - \bar{W}^{\cdot}(t). \tag{27}$$

Using (6) and (12) we have

$$\bar{W}^{\cdot}(t) = \int_{\Omega} \sigma(t) : \varepsilon^{\cdot}(t)\, dv + S(t),$$

$$S(t) = \int_{\Omega} \left[\frac{\partial Q}{\partial t}(t-\tau^*, -\tau^*) e(t) : e(t) + \int_0^t \frac{\partial^2 Q}{\partial t\, \partial \tau}(t-\tau^*, \tau-\tau^*)(e(t) - e(\tau)) : (e(t) - e(\tau))\, d\tau \right] dv. \tag{28}$$

Substitute these relations into the identity (27). With the help of (1), (10), (11) and Stokes' formula we find

$$\theta(t)\bar{H}^{\cdot}(t) = -S(t). \tag{29}$$

For adiabatic loading the Second Law of thermodynamics says that the entropy of the body increases monotonously. Thus from (29) it follows that

$$\frac{\partial Q}{\partial t}(t, \tau) \leqslant 0, \quad \frac{\partial^2 Q}{\partial t\, \partial \tau}(t, \tau) \leqslant 0, \quad (0 \leqslant \tau \leqslant t). \tag{30}$$

Relations (3) and (30) are the system of thermodynamic inequalities, which guarantee the correctness of the model for a viscoelastic medium.

For the aging elastic body ($Q(t, \tau) = 0$) inequalities (30) hold. For a nonaging viscoelastic material ($G = \text{const}$, $Q = Q_0(t-\tau)$) these inequalities may be written in the form $Q_0^{\cdot}(t) \leqslant 0$, $Q_0^{\cdot\cdot}(t) \geqslant 0$. Analogous conditions have been formulated by Dafermos (1970a,b).

6. FIRST-ORDER PHASE TRANSITIONS IN AGING VISCOELASTIC BODIES

Denote by θ_0 the temperature of the phase transition when the body and the surface forces are equal to zero. The body is in its natural state at the temperature θ_0 and occupies the domain Ω. The material is in two phases, the substance in the ith phase occupies the domain $\Omega_i(0)$. At the moment $t = 0$, body and surface forces are applied to the body. Due to their influence the substance in the first phase develops into the second phase. We assume that inverse phase transition is not realized. At the moment $t \geqslant 0$ material in the ith phase occupies the connected domain $\Omega_i(t)$. The domains $\Omega_i(t)$ are divided by the smooth surface $\gamma(t)$. We assume that $\gamma(t)$ does not cross the surface Γ and the domain $\Omega_2(t)$ lies in $\Omega_1(t)$. The behavior of the material in the ith phase before the phase transition is described by the constitutive equations of an homogeneous, aging, viscoelastic medium with constant bulk modulus K_i, current shear modulus $G_i(t)$ and relaxation measure $Q_i(t, \tau)$. According to (2) the free energy per unit mass of material in the ith phase $\Psi_i(t)$ is equal to

$$\Psi_i = \Psi_{i0} + W_i - H_{i0}(\theta-\theta_0) - c_i(\theta-\theta_0)^2/(2\theta_0),$$

$$\rho W_i = \tfrac{1}{2}K_i\bar{\varepsilon}^2(t) + G_i(t)\left[R_i(t,0)e(t):e(t) + \int_0^t \frac{\partial R_i}{\partial\tau}(t,\tau)(e(t)-e(\tau)):(e(t)-e(\tau))\,d\tau\right]. \quad (31)$$

where Ψ_{i0} and H_{i0} are the free energy and entropy per unit mass of the material in the ith phase in its natural configuration at the temperature θ_0, c_i is the heat capacity per unit mass.

The constitutive equations of the material can be written in the form

$$\bar{\sigma}_i(t) = 3K_i\bar{\varepsilon}(t), \quad s_i(t) = 2G_i(t)\left[e(t) - \int_0^t \frac{\partial R_i}{\partial\tau}(t,\tau)e(\tau)\,d\tau\right], \quad (32)$$

where $\bar{\sigma}_i = \sigma_i : I$, $s_i = \sigma_i - \frac{1}{3}\bar{\sigma}_i I$.

We propose the following expression for the free energy per unit mass of the substance in the second phase after the phase transformation

$$\Psi_2 = \Psi_{20} + W_2 - H_{20}(\theta-\theta_0) - c_2(\theta-\theta_0)^2/(2\theta_0),$$

$$\rho W_2(t) = \tfrac{1}{2}K_2(\bar{\varepsilon}(t)-\bar{\varepsilon}_*)^2 + G_2(t-\tau_*)\left[R_2(t-\tau_*,0)(e(t)-e_*):(e(t)-e_*)\right.$$

$$\left. + \int_{\tau_*}^t \frac{\partial R_2}{\partial\tau}(t-\tau_*,\tau-\tau_*)(e(t)-e(\tau)):(e(t)-e(\tau))\,d\tau\right]. \quad (33)$$

where $\bar{\varepsilon}_* = \varepsilon_* : I$, $e_* = \varepsilon_* - \frac{1}{3}\bar{\varepsilon}_* I$.

Here $\varepsilon_* = \varepsilon_*(\tau_*)$ is the strain tensor, which describes the deformation of the reference configuration into the natural configuration of the material in the second phase after the phase transition; $\tau_* = \tau_*(\xi)$ is the time when the phase transition occurs in the neighbourhood of the point ξ.

According to Arutyunyan *et al.* (1987) the constitutive equations of the material in the second phase after the phase transformation have the form

$$\bar{\sigma}_2(t) = 3K_2(\bar{\varepsilon}(t)-\bar{\varepsilon}_*),$$

$$s_2(t) = 2G_2(t-\tau_*)\left[(e(t)-e_*) - \int_{\tau_*}^t \frac{\partial R_2}{\partial\tau}(t-\tau_*,\tau-\tau_*)(e(\tau)-e_*)\,d\tau\right]. \quad (34)$$

Equalities (32) and (34) may be written as

$$\sigma_i(t) = \rho\frac{\partial W_i}{\partial\bar{\varepsilon}(t)}. \quad (35)$$

For first-order phase transitions the free energy per unit mass in the natural state is continuous: $\Psi_{10} = \Psi_{20}$ and the derivatives of the free energy have finite jumps (Landau and Lifshitz, 1976). For isotropic phase transformations the jump of the density is determined by the linear coefficient of compression λ and the jump of entropy is given by the latent heat of the plane transition μ: $H_{20} = H_{10} + \mu\theta_0^{-1}$.

Suppose that the body element, which was in equilibrium before the phase transformation under the action of external forces, is in equilibrium under the action of the same forces after the transformation and phase deformation occur:

$$\sigma_2(\tau_*) = \sigma_1(\tau_*), \quad \text{if } \bar{\varepsilon}_2(\tau_*) = \bar{\varepsilon}_1(\tau_*) - 3\lambda, e_2(\tau_*) = e_1(\tau_*).$$

An analogous assumption was proposed in Alts and Strahlow (1984) for the cooling process in glass and in Arutyunyan and Drozdov (1985) for the phase transition in elastic and viscoelastic media.

From this hypothesis and (32) and (34) we obtain

$$\bar{\varepsilon}_* = \left(1-\frac{K_1}{K_2}\right)\bar{\varepsilon}_1(\tau_*)-3\lambda,$$

$$e_* = \left(1-\frac{G_1(\tau_*)}{G_2(0)}\right)e_1(\tau_*)+\frac{G_1(\tau_*)}{G_2(0)}\int_0^{\tau_*}\frac{\partial R_1}{\partial\tau}(\tau_*,\tau)e_1(\tau)\,d\tau.$$

At $G_2(0)\neq 0$ these equalities define the natural configuration of the body element in the second phase. The identity $G_2(0)=0$ characterizes an elastic liquid (Lurie, 1980), for which all configurations with different shifts are equivalent.

The problem is comprised of the determination of the interface $\gamma(t)$, the stress field $\sigma(t)$ and the displacement field $u(t)$ when the body and surface forces and the heat exchange conditions on the boundary are given. In the case when large temperature gradients appear in the body (due to the heat exchange), the position of the interface may be found from the solution of the Stephan problem for the heat equation. Afterwards the stress and displacement fields are obtained by solving the contact problem. In the case when the temperature gradients are small enough the mechanical stresses have the essential influence on the phase transition and we need an additional condition to determine the interface.

7. VARIATIONAL PRINCIPLES IN VISCOELASTICITY WHICH ACCOUNT FOR PHASE TRANSITIONS

We first deal with the isothermal loading when the body temperature is constant and equal to θ_0. Fix the moment $t\geqslant 0$ and the deformation history $\{\gamma(\tau),u(\tau,\xi),(0\leqslant\tau\leqslant t)\}$. Denote by $U_*(t)$ the set of smooth surfaces $\gamma^0(t)$, which lie in Ω and divide this domain into connected subdomains $\Omega_i^0(t)$, and displacement fields $u^0(t,\xi)$, which are continuous in Ω, continuously differentiable in $\Omega_i^0(t)$ and which have finite jumps of derivatives across the surface $\gamma^0(t)$. Choose an admissible element $(\gamma^0(t),u^0(t,\xi))\in U_*(t)$. According to Roitburd (1971), we assume that the free energy and entropy of the interface can be neglected.

We formulate the following principle of minimum free energy of a system:

The real interface $\gamma(t)$ and the real displacement field $u(t,\xi)$ minimize the function

$$E(t)=\Psi(t)-A(t)=\rho\Psi_0|\Omega|+\bar{W}(t)-A(t),\quad \bar{W}(t)=\int_{\Omega_1^0(t)}\rho W_1(t)\,dv+\int_{\Omega_2^0(t)}\rho W_2(t)\,dv \tag{36}$$

on the set $U_*(t)$.

Consider the variation of the surface $\gamma(t)$ at which the domain $\Omega_i(t)$ transforms into $\Omega_i'(t)$. We assume that this transformation is described by the formula $r'=r+\delta w$, where r, r' are the radius vectors of points in the reference and in the perturbed configurations. The function $\delta w(t,\xi)$ is continuous in Ω and continuously differentiably in $\Omega_i(t)$. It has finite jumps across $\gamma(t)$ and vanishes on Γ. Denote by $\delta_0u(t,\xi)$ the variation of the function $u(t,\xi)$. The complete variation of the displacement field $u(t,\xi)$ is given by $\delta u(t)=\delta_0u(t)+\delta w(t)\cdot\nabla u(t)$. If $|\nabla u(t)|\ll 1$, then it may be shown that the complete variation of the strain field $\varepsilon(t,\xi)$ is defined by the equality $\delta\varepsilon(t)=\delta_0\varepsilon(t)+\delta w(t)\cdot\nabla\varepsilon(t)$, where $\delta_0\varepsilon(t)$ is connected with $\delta_0u(t)$ by (1).

From (1) and Stoke's formula it follows that

$$\delta\int_{\Omega_1(t)} V(\varepsilon(t))\,\mathrm{d}v = \int_\Gamma N\cdot\frac{\partial V}{\partial\varepsilon}(\varepsilon(t))\cdot\delta u(t)\,\mathrm{d}s - \int_{\gamma(t)} n(t)$$
$$\cdot\left\{\left[V(\varepsilon(t))I - \frac{\partial V}{\partial\varepsilon}(\varepsilon(t))\cdot\nabla u^T(t)\right]\cdot\delta w(t) + \frac{\partial V}{\partial\varepsilon}(\varepsilon(t))\cdot\delta u(t)\right\}\mathrm{d}s$$
$$-\int_{\Omega_1(t)}\left(\nabla\cdot\frac{\partial V}{\partial\varepsilon}(\varepsilon(t))\right)\cdot(\delta u(t)-\nabla u^T(t)\cdot\delta w(t))\,\mathrm{d}v \quad (37)$$

for any sufficiently smooth function $V(\varepsilon)$. Here $n(t)$ is the unit normal to $\gamma(t)$ pointing into $\Omega_1(t)$. An analogous formula can be derived for

$$\delta\int_{\Omega_2(t)} V(\varepsilon(t))\,\mathrm{d}v.$$

Using (8), (35)–(37), we find

$$\delta E(t) = \int_\Gamma (N\cdot\sigma_1(t)-f(t))\cdot\delta u(t)\,\mathrm{d}s + \int_{\gamma(t)} n(t)\cdot\{[\rho(W_2(t)-W_1(t))I$$
$$-(\sigma_2(t)\cdot\nabla u_2^T(t)-\sigma_1(t)\cdot\nabla u_1^T(t))]\cdot\delta w(t)+(\sigma_2(t)-\sigma_1(t))\cdot\delta u(t)\}\,\mathrm{d}s$$
$$-\int_\Omega(\nabla\cdot\sigma(t)+\rho F(t))\cdot(\delta u(t)-\nabla u^T(t)\cdot\delta w(t))\,\mathrm{d}v. \quad (38)$$

From (38), the principle of minimum free energy and the arbitrariness in choice of $\delta u(t)$, $\delta w(t)$ we obtain the equilibrium equation (10), the boundary condition (11), the boundary condition on the interface $\gamma(t)$

$$n(t)\cdot\sigma_2(t) = n(t)\cdot\sigma_1(t), \quad (\xi\in\gamma(t)) \quad (39)$$

and the energy ratio on the surface $\gamma(t)$

$$\rho[W_2(t)-W_1(t)]n(t) = n(t)\cdot[\sigma_2(t)\cdot\nabla u_2^T(t)-\sigma_1(t)\cdot\nabla u_1^T(t)], \quad (\xi\in\gamma(t)).$$

According to Berdichevskii (1983a), one can show that this identity reduces to

$$\rho[W_2(t)-W_1(t)] = n(t)\cdot\sigma(t)\cdot[\nabla u_2^T(t)-\nabla u_1^T(t)]\cdot n(t), \quad (\xi\in\gamma(t)). \quad (40)$$

Let us now consider the case of adiabatic loading, when there is no heat flow across the surface of the body. For simplicity we assume that the heat capacity does not change during phase transition: $c_1 = c_2 = c$. Fix the moment $t \geqslant 0$ and the deformation history $\{\theta(\tau), \gamma(\tau), u(\tau,\xi), (0\leqslant\tau\leqslant t)\}$. Denote by $B_*(t)$ the set of admissible temperature values $\theta^0(t)$, admissible interfaces $\gamma^0(t)$ and admissible displacement fields $u^0(t,\xi)$. Choose the element $(\theta^0(t),\gamma^0(t),u^0(t,\xi))\in B_*(t)$. The internal energy and the entropy of the body are given by

$$\Phi(t) = \rho[(\Psi_{10}+H_{10}\theta_0)+c((\theta^0(t))^2-\theta_0^2)/(2\theta_0)]|\Omega| + \rho\mu|\Omega_2^0(t)| + \bar{W}(t),$$
$$\bar{H}(t) = \rho[H_{10}+c(\theta^0(t)-\theta_0)/\theta_0]|\Omega| + \rho\mu|\Omega_2^0(t)|/\theta_0,$$
$$|\Omega_i^0(t)| = \int_{\Omega_i^0(t)}\mathrm{d}v. \quad (41)$$

We propose the following formulation of Gibbs' principle:

The real temperature $\theta(t)$, the real interface $\gamma(t)$ and the real displacement field $u(t,\xi)$ maximize the entropy $\bar{H}(t)$ of the body on the subset of $B_*(t)$, whose elements satisfy the conservation law of energy (17).

From (17), (36) and (41) it follows that

$$\theta(t)=\theta_-(t)\left\{1+\frac{2\theta_0}{\rho c|\Omega|\theta_-^2(t)}[(E_-(t)-E(t))-\rho\mu(|\Omega_2(t)|-|\Omega_{2-}(t)|)]\right\}^{1/2}. \tag{42}$$

According to Stoke's formula we have

$$\delta|\Omega_2(t)|=\int_{\gamma(t)} n(t)\cdot\delta w(t)\,\mathrm{d}s.$$

Using this relation and (41), (42) we find

$$\theta(t)\delta\bar{H}(t)=\rho\mu\frac{\theta(t)-\theta_0}{\theta_0}\int_{\gamma(t)} n(t)\cdot\delta w(t)\,\mathrm{d}s-\delta E(t). \tag{43}$$

From (38), (43) and Gibbs' principle we obtain the equilibrium equation (10), the boundary conditions (11), (39) and the energy condition on the interface

$$\rho\mu\frac{\theta(t)-\theta_0}{\theta_0}n(t)=\rho[W_2(t)-W_1(t)]n(t)-n(t)\cdot\sigma(t)\cdot[\nabla u_2^T(t)-\nabla u_1^T(t)],\quad(\xi\in\gamma(t)). \tag{44}$$

This relation reduces to the scalar equality

$$\rho\mu\frac{\theta(t)-\theta_0}{\theta_0}=\rho[W_2(t)-W_1(t)]-n(t)\cdot\sigma(t)\cdot[\nabla u_2^T(t)-\nabla u_1^T(t)]\cdot n(t),\quad(\xi\in\gamma(t)). \tag{45}$$

We now calculate the time derivatives of the internal energy and the entropy of the body. According to (41) we get

$$\begin{aligned}\bar{\Phi}^{\cdot}(t)&=\rho c\theta(t)\theta^{\cdot}(t)|\Omega|/\theta_0+\rho\mu|\Omega_2(t)|^{\cdot}+\bar{W}^{\cdot}(t),\\ \bar{H}^{\cdot}(t)&=\rho[c\theta^{\cdot}(t)|\Omega|+\mu|\Omega_2(t)|^{\cdot}]/\theta_0.\end{aligned} \tag{46}$$

Let us suppose that during the time Δt the domain $\Omega_t(t)$ transforms into $\Omega_t(t+\Delta t)$. We assume that this transformation is described by the equation $r'=r+\Delta w$, where the function $\Delta w(t,\xi)$ has the same properties as $\delta w(t,\xi)$. Denote by $w^{\cdot}(t)$ the limit of the ratio $\Delta w(t)/\Delta t$ as $\Delta t\to 0$. Similarly to (38) we obtain

$$\begin{aligned}\bar{W}^{\cdot}(t)=&\int_\Gamma N\cdot\sigma_1(t)\cdot u^{\cdot}(t)\,\mathrm{d}s+\int_{\gamma(t)} n(t)\cdot\{[\rho(W_2(t)-W_1(t))I\\ &-(\sigma_2(t)\cdot\nabla u_2^T(t)-\sigma_1(t)\cdot\nabla u_1^T(t))]\cdot w^{\cdot}(t)+(\sigma_2(t)-\sigma_1(t))\cdot u^{\cdot}(t)\}\,\mathrm{d}s\\ &-\int_\Omega(\nabla\cdot\sigma(t))\cdot(u^{\cdot}(t)-\nabla u^T(t)\cdot w^{\cdot}(t))\,\mathrm{d}v+S(t),\end{aligned} \tag{47}$$

where $u^{\cdot}(t)$ is the complete derivative of $u(t)$ and $S(t)$ is defined by the formula

$$S(t) = \int_{\Omega_1(t)} \left[\frac{\partial Q_1}{\partial t}(t,0)e(t):e(t) + \int_0^t \frac{\partial^2 Q_1}{\partial t\,\partial \tau}(t,\tau)(e(t)-e(\tau)):(e(t)-e(\tau))\,d\tau \right] dv$$
$$+ \int_{\Omega_2(0)} \left[\frac{\partial Q_2}{\partial t}(t,0)e(t):e(t) + \int_0^t \frac{\partial^2 Q_2}{\partial t\,\partial \tau}(t,\tau)(e(t)-e(\tau)):(e(t)-e(\tau))\,d\tau \right] dv$$
$$+ \int_{\Omega_2(t)\,\Omega_2(0)} \left[\frac{\partial Q_2}{\partial t}(t-\tau_*,0)(e(t)-e_*):(e(t)-e_*) \right.$$
$$\left. + \int_{\tau_*}^t \frac{\partial^2 Q_2}{\partial t\,\partial \tau}(t-\tau_*,\tau-\tau_*)(e(t)-e(\tau)):(e(t)-e(\tau))\,d\tau \right] dv.$$

From (46), (47), the First Law of thermodynamics (26) and equalities (10), (11), (39) and (44) it follows that

$$\theta(t)\bar{H}^*(t) = -S(t). \tag{48}$$

Equality (48) denotes the Second Law of thermodynamics for a viscoelastic medium at adiabatic phase transitions. Relations (10), (11), (32), (34), (39), (45) and (48) are the complete system of equations for the description of the stresses in a viscoelastic body. For the aging elastic medium $S(t) = 0$ and from (41), (45) and (48) we get

$$\frac{\rho\mu^2}{c\theta_0}\frac{|\Omega_2(t)/\Omega_2(0)|}{|\Omega|} = \rho[W_1(t) - W_2(t)] - n(t)\cdot\sigma(t)\cdot[\nabla u_1^T(t) - \nabla u_2^T(t)]\cdot n(t), \quad (\xi \in \gamma(t)). \tag{49}$$

Equation (49) defines the interface position at adiabatic phase transitions in the aging elastic body. In the case when the latent heat of transformation is neglected ($\mu \to 0$) or when the dimensions of the domain, which the substance occupies in the new phase, are essentially less than the dimensions of the body ($|\Omega_2(t)/\Omega_2(0)|/|\Omega| \to 0$) this equation reduces to (40).

8. DEVELOPMENT OF A NUCLEUS OF MELT IN AN AGING VISCOELASTIC MEDIUM

Consider the melting of a viscoelastic body at isothermal loading. The body is heated to the phase transition temperature θ_0 and is in its natural state. At the first stage the melting process consists of the growth of liquid phase nuclei. If the number of nuclei is not too large and their mutual influence is neglected we may study the development of an isolated liquid phase nucleus in an infinite medium. Assume that the nucleus occupies the spherical domain with radius a_0. Introduce the spherical system of coordinates (r, ϑ, φ), which origin lies in the centre of the sphere. At the moment $t = 0$ the pressure $q = q(t)$ is applied to the medium at infinity, $(q(0) = 0,\ q^*(t) > 0)$. Under the action of the external load dimensions of the nucleus increase. Denote by $a(t)$ the nucleus radius at the moment $t \geqslant 0$.

For simplicity we consider the growth of the nucleus in the incompressible viscoelastic medium. The creation of a nucleus of solid phase in a compressible elastic body and the stability of this nucleus was studied in Johnson and Voorhees (1987) and Strehlow (1988).

Suppose that the behaviour of the material in the solid phase is governed by the constitutive equations of an incompressible homogeneous aging viscoelastic body

$$\rho W_1(t) = G(t)\left[R(t,0)e(t):e(t) + \int_0^t \frac{\partial R}{\partial \tau}(t,\tau)(e(t)-e(\tau)):(e(t)-e(\tau))\,d\tau \right],$$
$$\bar{\sigma}(t) = -3p(t), \quad s(t) = 2G(t)\left[e(t) - \int_0^t \frac{\partial R}{\partial \tau}(t,\tau)e(\tau)\,d\tau \right],$$
$$\bar{\varepsilon}(t) = 0, \quad (a(t) \leqslant r < \infty). \tag{50}$$

where p is the pressure. The mechanical behaviour of material in liquid phase obeys the constitutive laws of an incompressible elastic fluid

$$\rho W_2(t) = 0, \quad \bar{\sigma}(t) = -3p(t), \quad s(t) = 0, \quad \bar{\varepsilon}(t) = 0, \quad (0 \leqslant r \leqslant a_0);$$
$$\bar{\varepsilon}(t) = -3\lambda, \quad (a_0 < r < a(t)). \tag{51}$$

We obtain expressions (50) and (51) in the limit of (31)–(34) as $K_1, K_2 \to \infty$.

Denote by $u(t, r)$ the radial displacement of the body points. Write the incompressibility conditions as

$$u' + 2ur^{-1} = 0, \quad (0 \leqslant r \leqslant a_0, a(t) \leqslant r < \infty); \quad u' + 2ur^{-1} = -3\lambda, \quad (a_0 < r < a(t)),$$

where $u' = \partial u/\partial r$.

Integrate these equations by r. From the identity of the radial displacement at $r = a_0$ and $r = a(t)$ we get

$$u = 0, \quad (0 \leqslant r \leqslant a_0); \quad u = \lambda(a_0^3 - r^3)r^{-2}, \quad (a_0 < r < a(t));$$
$$u = \lambda(a_0^3 - a^3(t))r^{-2}, \quad (a(t) \leqslant r < \infty). \tag{52}$$

Substitute these expressions into (50) and (51) and find the physical components of the stress tensor σ

$$\sigma_r = \sigma_\vartheta = \sigma_\varphi = -p(t), \quad (0 \leqslant r < a(t));$$
$$\sigma_r = -p(t) - 4\lambda L(t)r^{-3}, \quad \sigma_\vartheta = \sigma_\varphi = -p(t) + 2\lambda L(t)r^{-3}, \quad (a(t) \leqslant r < \infty),$$
$$L(t) = G(t)\left[(a_0^3 - a^3(t)) - \int_0^t \frac{\partial R}{\partial \tau}(t, \tau)(a_0^3 - a^3(\tau))\, d\tau\right]. \tag{53}$$

Integrate the equilibrium equation $\sigma_r' + 2(\sigma_r - \sigma_\vartheta)r^{-1} = 0$ with respect to r from $a(t)$ to infinity. Using (53) and the boundary condition $\sigma_r(t, \infty) = -q(t)$, we have

$$\sigma_r(t, a(t)) = -q(t) - 4\lambda L(t)a^{-3}(t). \tag{54}$$

Substitute (50)–(52), (54) into (40)

$$\sigma G(t)\lambda^2 a^{-6}(t)\left[R(t, 0)(a^3(t) - a_0^3)^2 + \int_0^t \frac{\partial R}{\partial \tau}(t, \tau)(a^3(t) - a^3(\tau))^2\, d\tau\right]$$
$$= -3\lambda\left\{q(t) - 4G(t)\lambda a^{-3}(t)\left[(a^3(t) - a_0^3) - \int_0^t \frac{\partial R}{\partial \tau}(t, \tau)(a^3(\tau) - a_0^3)\, d\tau\right]\right\}.$$

Introducing the dimensionless values $G_*(t) = G(t)/G(0)$, $q_*(t) = q(t)/(2G(0)\lambda)$, $a_*(t) = a(t)/a_0$, $z(t) = a_*^6(t)$, we write this equation in the following form

$$\left[1 - \frac{q_*(t)}{G_*(t)}\right]z(t) = R(t, 0) - \int_0^t \frac{\partial R}{\partial \tau}(t, \tau)z(\tau)\, d\tau. \tag{55}$$

Equation (55) is a linear Volterra equation for the function $z(t)$. After solving this equation we find the interface radius from the identity $a(t) = a_0 z^{1/6}(t)$ and the displacements and stresses from (52) and (53).

For a nonaging elastic medium ($G = \text{const}$, $R(t, \tau) = 1$) we obtain

$$z(t) = [1 - q(t)/(2G\lambda)]^{-1}. \tag{56}$$

According to (56) the dimensions of the nucleus of the melt increase as the pressure grows and tend to infinity as $q(t)$ approaches $2G\lambda$.

For an aging elastic medium ($G = G(t), Q(t, \tau) = 0$) equality (55) reduces to the ordinary differential equation $[G_*(t) - q_*(t)]z^* = q_*^*(t)z$, $z(0) = 1$. The solution of this equation is

$$z(t) = \exp\left\{\int_0^t q^*(\tau)[2G(\tau)\lambda - q(\tau)]^{-1}\,d\tau\right\}. \tag{57}$$

From (57) it follows that the material aging causes the diminution of the dimensions of the nucleus. Note that the radius of the nucleus depends on the loading history. A discussion of this phenomenon can be found in Gorokh and Arkharov (1989).

Consider the standard viscoelastic body (Arutyunyan and Kolmanovskii, 1983) whose behaviour is governed by the equation $s^* + \gamma s = 2(Ge^* + \gamma G_0 e)$, where G and G_0 are the current and the limit shear modulus, γ^{-1} is the characteristic time of relaxation. Equation (55) reduces to the differential equation

$$[1 - q_*(t)]z^* = [q_*^*(t) + q_*(t) - \kappa]z + \kappa, \quad z(0) = 1, \quad (z^* = dz/dt_*, t_* = \gamma t, \kappa = G_0/G < 1).$$

Suppose that the pressure tends towards the limit value q^0 and the characteristic time of change of the external forces is essentially less than the characteristic time of relaxation. In this case we may regard q_* as a constant, which is equal to $q_*^0 = q^0/(2G\lambda)$. The limit value of $z(t)$ as $t \to \infty$ is defined by

$$z^0 = [1 - q^0/(2G_0\lambda)]^{-1}. \tag{58}$$

From (56) and (58) it follows that at constant load the limit radius of the nucleus of the melt in a viscoelastic medium coincides with the radius of the nucleus in an elastic medium, whose shear modulus is equal to the limit shear modulus of a viscoelastic material.

If q^0 satisfies the inequality $2G_0\lambda < q^0 < 2G\lambda$, then the dimensions of the nucleus in an elastic medium stay finite and the radius of the nucleus in a viscoelastic medium tends to infinity as $t \to \infty$.

The characteristic time for establishing thermodynamic equilibrium at the melting of viscoelastic body $T = (1 - q_*^0)[\gamma(\kappa - q_*^0)]^{-1}$ depends on the intensity of pressure and tends to infinity as q^0 approaches the critical value $2G_0\lambda$.

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